POLYHEDRA

A polyhedron is a three-dimensional solid bounded by a finite number of polygonal *faces* which are parts of planes. The faces meet in pairs along *edges* which are straight line segments, and the edges meet in points called *vertices*. For simplicity we will only look at convex polyhedra — this means that for any two points in the solid, the straight line segment joining them is also contained in the solid. In making this restriction we are ruling out many beautiful examples such as stellated polyhedra, but life will be a little simpler!

Your tasks: Produce a poster explaining one or several of the following:

1. **The Platonic Solids** A Platonic solid is a convex regular polyhedron (regular means that each face is a regular polygon, every face is congruent to every other face, and the angle between any two adjoining faces is the same). There are five such figures: The tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. These were known to the ancient greeks, indeed they are named after Plato who theorized that the basic elements (earth, air, fire and water) were made from them: Air from the octahedron, Fire from the tetrahedron, Earth from the cube, and water from the icosahedron. The dodecahedron he reserved for the heavens.

   Apart from their mystical properties, the greeks knew quite a lot about the mathematical properties of the regular polyhedra: Theaetetus (ca. 417 BC - 369 BC) was probably the first to prove that there are only five of them, and also explored their geometry by computing the ratios of edge lengths to diameters of inscribed and circumscribed spheres. Much of Theaetetus's work has come down to us through Euclid, who included it in the last (thirteenth) of the books of his *elements*.

   Explain why there are only five regular polyhedra, based on the following fact: For any convex polyhedron, at each vertex the sum of the angles which the neighbouring faces make at that vertex is less than 2π radians (or 360 degrees). Show how to derive an expression for the angles between neighbouring faces.

2. **The Archimedean Solids** An Archimedean solid is a 'semi-regular' convex polyhedron. This means that it is composed of two or more types of regular polygons meeting in identical vertices. As for the Platonic solids we require each of the faces to be a regular polygon, though the faces do not all have to be congruent — there can be several different regular polygons appearing as edges. However, we require that all of the vertices be identical: Given any two vertices, there is some rotation which takes one to the other and rotates the entire solid into itself. This means in particular that at each vertex there are the same kinds of faces meeting, in the same order.

   The Archimedean solids take their name from Archimedes, who discussed them in a now-lost work. During the Renaissance, artists and mathematicians valued pure forms and rediscovered the Archimedean solids, with the last ones being rediscovered by Johannes Kepler in 1619 (he was so interested in polyhedra that he tried to make them the basis for a model of the solar system, explaining the spacing between the planetary orbits by the sizes of a nested family of Platonic solids). There are thirteen Archimedean solids, as well as two infinite families called the *prisms* and *antiprisms*. The prisms are obtained by taking two copies of a regular

n-sided polygon (n-gon), and joining each edge of one to an edge of the other with a square. The antiprisms again take two copies of a n-gon, but join each edge of one to a vertex of the other using an equilateral triangle.

Since all vertices are identical, each Archimedean solid can be described by listing the polygons which meet each vertex in order (say, reading clockwise). Thus a prism would correspond to the sequence $(n, 4, 4)$, since around each vertex we have the n-gon, then a square, then a square (we could also write $(4, n, 4)$ or $(4, 4, n)$, corresponding to choosing a different face to start the description — the list is cyclic, so we understand that it 'wraps around' from the last term to the first one). Similarly, an antiprism is described by $(n, 3, 3, 3)$ (or $(3, n, 3, 3)$ or $(3, 3, n, 3)$ or $(3, 3, 3, n)$), and a tetrahedron by $(3, 3, 3)$.

The truncated tetrahedron, obtained by cutting each vertex off a third of the way along each of the edges it meets, has faces which are hexagons and triangles, appearing in the sequence $(3, 6, 6)$. The others are: The cuboctahedron $(3, 4, 3, 4)$, the truncated cube $(3, 8, 8)$, the truncated octahedron $(4, 6, 6)$, the rhombicuboctahedron $(3, 4, 4, 4)$, the truncated cuboctahedron $(4, 6, 8)$, the snub cube $(3, 3, 3, 3, 4)$, the icosidodecahedron $(3, 5, 3, 5)$, the truncated dodecahedron $(3, 10, 10)$, the truncated icosahedron (or 'buckyball') $(5, 6, 6)$, the rhombicosidodecahedron $(3, 4, 5, 4)$, the truncated icosidodecahedron $(4, 6, 10)$, and the snub dodecahedron $(3, 3, 3, 3, 5)$.

Explain why there are only thirteen Archimedean solids apart from the prisms and antiprisms, based on the following facts: List the adjoining sides near each vertex as $(p_1, p_2, \ldots, p_q)$.

- *(a) There must be at least three faces meeting each vertex:* $q \geq 3$;
- *(b) Each face must have at least 3 sides:* $p_i \geq 3$ *for* $i = 1, 2, \ldots, q$;
- *(c) The sum of the interior angles at each vertex must be less than a full circle:*

$$\left(\frac{1}{2} - \frac{1}{p_1}\right) + \left(\frac{1}{2} - \frac{1}{p_2}\right) + \ldots + \left(\frac{1}{2} - \frac{1}{p_q}\right) < 1;$$

- *(d) if any of the faces has an odd number* k *of sides, then either* k *must appear in the description at least 3 times, or the description must contain a sequence of the form* x, k, x *(i.e. at least one of the* k*-gons meeting that vertex must be bordered by two identical polygons). To see why this is so, imagine one of the* k*-gon faces, and suppose that as we follow around its edges in clockwise order the adjoining faces have* $j_1, j_2, \ldots, j_k$ *sides. Since all of the vertices are identical, each of the sequences* j_1, k, j_k *and* j_2, k, j_1 *and so on, up to* j_k, k, j_{k-1} *must appear in the description somewhere. If* k *appears only once in the description, then all of these sequences must be the same,* $j_1 = j_2 = \ldots = j_k$ *and there is a sequence of the form* (x, k, x). *If* k *appears twice, then either all of the* j*'s are equal (so there is a sequence* x, k, k, x*), or we must have* $j_1 = j_3 = \ldots = j_{k-1}$, *and* $j_2 = j_4 = \ldots = j_k$, *so* k *is even. In any other case* j *appears at least three times.*

[For example, (c) tells you that there can be at most 5 faces meeting each vertex, since the internal angle of any regular polygonal face is at least $\pi/3$, *so six faces would give angle at least* 2π*]*

3. **The Euler relation**

   Leonhard Euler proved a famous formula relating the number of edges, faces and vertices of a polyhedron: $F - E + V = 2$. You can easily check this for all of the simplest examples such as the Platonic or Archimedean solids, but Euler proved it for *all* convex polyhedra. In fact it is also true for non-convex polyhedra, as long as they do not have any 'holes'.

   Explain why the Euler relation holds, based on the following:

   (a) The Euler relation holds for a tetrahedron;

   (b) Moving any vertex without crossing edges does not change F, E or V;

   (c) Adding a vertex in the middle of a face with n edges increases F by $n-1$, V by 1, and E by n;

   (d) Any convex polyhedron can be 'built' by starting with a tetrahedron, adding enough vertices and then moving them around.

4. **The Descartes angle defect formula**

   Closely related to the Euler relation is the following remarkable formula, due to Descartes: At each vertex of a convex polyhedron, the *angle defect* is the difference between a full circle (2π radians or 360 degrees) and the sum of the angles made by the neighbouring faces at that vertex. Thus the angle defect is the angle that must be cut out of a flat sheet of paper in order to fold the part of the polyhedron near that vertex. The descartes formula says that the sum of all of the angle defects at all of the vertices is equal to 4π (or 720 degrees).

   Deduce the Descartes angle formula from the Euler relation, using the following facts:

   (a) The sum of all of the angles in any triangle is π;

   (b) Adding a vertex in the centre of each face does not change the total angle defect, since the total angle around the new vertex is 2π. If this is done in the middle of every face, the result is a polyhedron with all faces equal to triangles;

   (c) If every face of a polyhedron is triangular, then $3F = 2E$;

   (d) The total angle defect is equal to $2\pi V$ minus the sum of all of the internal angles of all of the faces;

   (e) The Euler relation: $V - E + F = 2$

Posters will be assessed for originality, creative graphics, overall appeal and, of course, correctness and elegance of the mathematics. Scores will contribute to the school's aggregate for the day and there will be an award for best poster overall. Don't forget to include the school's name somewhere in the poster. If you wish to provide any mathematical explanations or justifications in relation to any of your answers, you may include these on separate paper or send them by email prior to the competition.